# First-order Policy Optimization for Robust Markov Decision Process


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Joint work with George Lan, Tuo Zhao

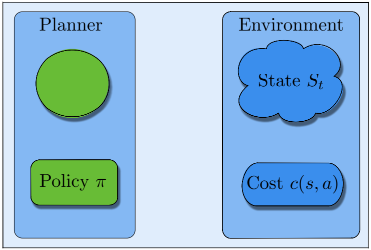
# Markov Decision Process & Policy Optimization

# Markov Decision Process

▷ **Sequential decision making over multiple timesteps ..**

**Key elements**

- policy $\pi$
- finite state space: $\mathcal{S}$
- finite action space: $\mathcal{A}$
- cost function $c$
- transition kernel $\mathbb{P}$



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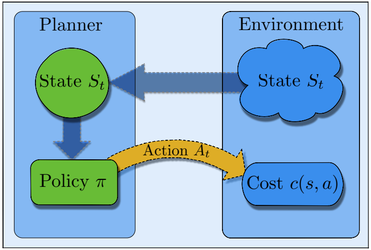


## Decision making:

1. Observe current state $S_t$ and feed into policy
2. Make $A_t$ following distribution $\pi(\cdot|S_t)$

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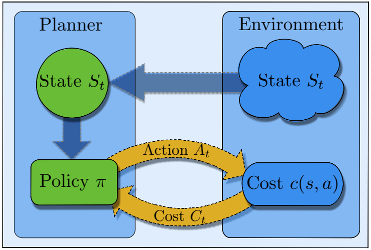


**Observing loss:** $C_t = c(S_t, A_t) \in [0, 1]$

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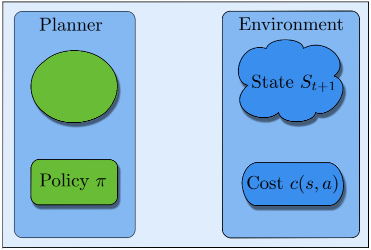


**State transition:** $S_{t+1}$ follows distribution $\mathbb{P}(\cdot|S_t, A_t)$

**Repeat decision process ..**

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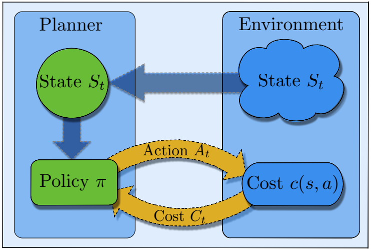


**Trajectory:**

$$\{(S_0, A_0, C_0), (S_1, A_1, C_1), \ldots, (S_t, A_t, C_t), \ldots\}$$

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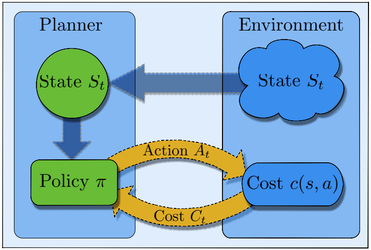


**Performance (value function):**

$$V_{\mathbb{P}}^{\pi}(s) = \mathbb{E}_{\mathbb{P}}^{\pi}\left[\sum_{t=0}^{\infty} \underbrace{\gamma^t C_t}_{\text{discounting future}} \,\middle|\, S_0 = s\right]$$

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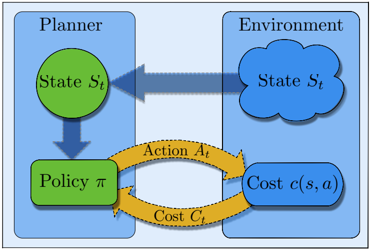


**Planning: find the optimal policy of**

$$\min_{\pi} V_{\mathbb{P}}^{\pi}(s), \ \forall s \in \mathcal{S}$$

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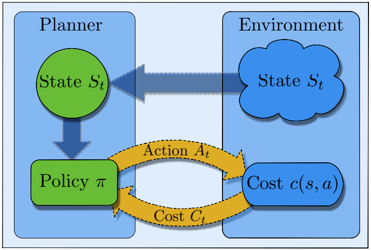



**Planning with an equivalent objective:**

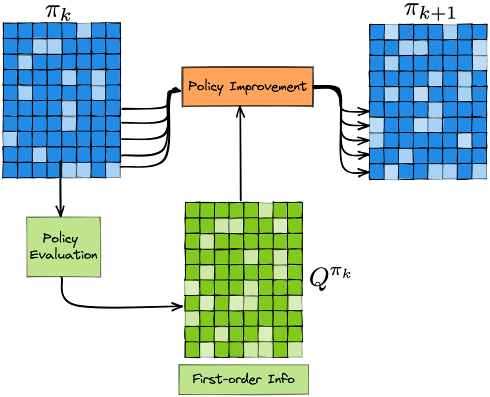
$$\min_{\pi} f_\rho(\pi) = \sum_{s\in\mathcal{S}} \rho(s) V_{\mathbb{P}}^{\pi}(s) \quad \Rightarrow \quad \text{Non-convex}$$

# Planning Methods for MDP

1. Linear programming based methods
   - stochastic primal-dual methods
2. Dynamic programming based methods
   - stochastic value iteration or Q-Learning
   - can diverge even with linear approximation
3. Nonlinear programming based methods
   - **policy gradient methods**
   - much more friendly to function approximation
   - Only until very recently, these methods were shown to exhibit comparable or even superior performance guarantees than alternative methods

## Policy Gradients – Overview

# Policy Gradients - A Basic Skeleton




## First-order policy optimization:

1. $\text{Eval}(\pi_k) \to Q_{\mathbb{P}}^{\pi_k}$
2. Construct gradient information $G_k$
3. $\text{Update}(\pi_k, G_k) \to \pi_{k+1}$
4. Repeat ..

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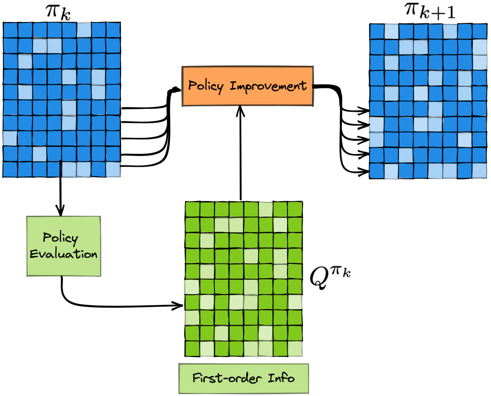


## Q-function:

$$Q^{\pi}_{\mathbb{P}}(s,a) = \mathbb{E}^{\pi}_{\mathbb{P}}\left[\sum_{t=0}^{\infty} \gamma^t c(S_t, A_t) \middle| S_0 = s, A_0 = a\right]$$

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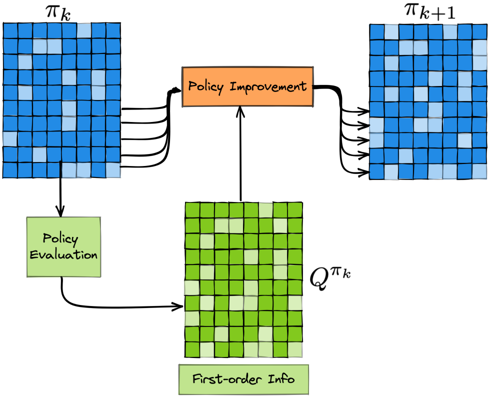


⋆ **Challenges:**

- Non-convex landscape
- Transition $\mathbb{P}$ and cost $c(\cdot)$ can be unknown

# Policy Gradients – Existing Development

1. Deterministic setting: exact first-order information:
   - Even-Dar, Kakade, Mansour '09: $\mathcal{O}(1/\sqrt{T})$ regret
   - Agarwal, Kakade, Lee, Mahajan '19: $\mathcal{O}(1/T)$
   - Cen et. al. '20: linear for entropy regularized MDPs

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   - Agarwal, Kakade, Lee, Mahajan '19: $\mathcal{O}(1/\epsilon^4)$
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3. Policy mirror descent (Lan, '21)
   - Deterministic: linear for both regularized and un-regularized
   - Stochastic: $\mathcal{O}(1/\epsilon^2)$ un-regularized; $\mathcal{O}(1/\epsilon)$ regularized

# Robust Markov Decision Process

# Motivating Examples

**I: Planning with Pre-collected Data $\mathcal{D}$**

## Direct approach

1. Estimate transition kernel $\widehat{\mathbb{P}} \approx \mathbb{P}$ from $\mathcal{D}$
2. Planning with estimated $\widehat{\mathbb{P}}$

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**Subject to randomness in data collection**

### Robust approach

1. Construct $\mathcal{P}$ s.t. $\mathbb{P} \in \mathcal{P}$ with high probability
2. Planning within $\mathcal{P}$ to hedge against randomness

## Motivating Examples

### II: Sim-to-real Robot Training

- Training environment (simulation) has $\mathbb{P}_{\text{sim}}$
- Deployment (real-life) environment has $\mathbb{P}_{\text{real}} \approx \mathbb{P}_{\text{sim}}$
- Ultimate goal is to perform well for $\mathbb{P}_{\text{real}}$

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### Robust approach

1. Construct $\mathcal{P}$ based on robustness preference
   - $\epsilon$-contamination model (Huber, '64):

$$\mathcal{P} = \{(1-\epsilon)\mathbb{P}_{\text{sim}} + \epsilon\mathbb{Q} : \mathbb{Q} \in \mathcal{Q} \text{ (pre-specified)}\}$$

   - Large $\epsilon$ yields stronger robustness
2. Planning within $\mathcal{P}$ to hedge against environment changes
   - Use only samples from interacting with $\mathbb{P}_{\text{sim}}$

# Robust Markov Decision Process

## ▷ Robust Objective:

$$\min_{\pi} \left\{ f_r(\pi) := \sum_{s\in\mathcal{S}} \rho(s) \underbrace{\max_{u\in\mathcal{U}} V^{\pi}_{\mathbb{P}_u}(s)}_{V^{\pi}_r(s)} \right\}$$

- $\mathbb{P}_u(\cdot|s,a) = \mathbb{P}_{\mathrm{N}}(\cdot|s,a) + u(\cdot|s,a)$ for $(s,a) \in \mathcal{S}\times\mathcal{A}$
- $\mathbb{P}_{\mathrm{N}}$: nominal transition kernel
- $\mathcal{U}$: index set for transition kernels (ambiguity set)

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## ▷ Structure of Ambiguity Set:

1. $(\mathrm{s}, \mathrm{a})$-rectangularity [our focus]:

$$\mathcal{U} = \Pi_{(s,a) \in \mathcal{S} \times \mathcal{A}} \, \mathcal{U}_{s,a}$$

- No coupling of uncertainties for different state-action pair
- Certain equivalence to nested robust formulation

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2. Popular alternative: $\mathrm{s}$-rectangularity
3. General cases: NP hard

# Robust Markov Decision Process

**Can we learn robust policy, while only given (stochastic) access to $\mathbb{P}_{\mathrm{N}}$?**

▷ **"Access of $\mathbb{P}_{\mathrm{N}}$"**

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## ▷ Existing Development

1. Value based methods (vast majority):
   - Tamar et. al, '14; Roy et. al, '17; Zhou et. al, '21; many others
2. Policy gradient methods (relatively few):
   - Wang and Zou, '22: smoothing argument
     - $\mathcal{O}(1/\epsilon^3)$ iterations in deterministic setting
     - $\mathcal{O}(1/\epsilon^7)$ samples in stochastic setting
     - Tailors to special $(s, a)$-rectangular set
   - Clearly not optimal (even $\mathcal{U} = \{\mathbf{0}\}$)

## Robust Policy Mirror Descent: Preview

# Preview of Results

▷ **Robust Policy Mirror Descent**

**Algorithm** RPMD update: $\pi_k \to \pi_{k+1}$

**Input**: Compute robust $Q_r^{\pi_k} := \max_{u \in \mathcal{U}} Q_{\mathbb{P}_u}^{\pi_k}$

**Update**: For every state $s \in \mathcal{S}$:

$$\pi_{k+1}(\cdot|s) = \operatorname{argmin}_{p \in \Delta_{\mathcal{A}}} \eta_k \langle Q_r^{\pi_k}(s, \cdot), p \rangle + \mathcal{D}_{\pi_k}^p(s)$$

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## ▷ Parameters and Variants

- $\eta_k$ – stepsize
- $\mathcal{D}_{\pi_k}^p(s) = w(p) - w(\pi_k(\cdot|s)) - \langle \nabla w(\pi_k(\cdot|s)), p - \pi_k(\cdot|s) \rangle$
  1. $w(\cdot)$: distance generating function (many choices)
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     $$\pi_{k+1}(a|s) \propto \pi_k(a|s) \exp\left(-\eta_k Q_r^{\pi_k}(s,a)\right)$$
  4. Tsallis divergence with index $q \in (0,1)$: $w(p) = -\sum_{a \in \mathcal{A}} p_a^p$
     - $\pi_{k+1}$ can be computed using simple bisection (Li and Lan, '23)

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1. **Versatile:** recovers PMD for non-robust MDP (Lan, '21)
2. **Efficient:**
   - Deterministic setting (exact $Q_r^{\pi_k}$): $\mathcal{O}(\log(1/\epsilon))$ iterations
   - Stochastic setting (estimated $Q_r^{\pi_k}$): $\mathcal{O}(1/\epsilon^2)$ samples
   - Optimal dependence on $\epsilon$

# First-order Viewpoint and Intuitions

# Issues with Policy Gradients

▷ **Not-so-friendly Landscape**

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## ▷ Additional Issues

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   - Special case discussed in Wang and Zou, '21
   - Local-to-global conversion already non-optimal in non-robust case

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⋆ **Need alternative first-order information** ⋆

# "Useful" First-order Information

**⋆ Robust Q-function as "Subgradient" ⋆**

▷ **Local Improvement**

$$V_r^{\pi'}(s) - V_r^{\pi}(s) \leq \frac{1}{1-\gamma}\mathbb{E}_{s' \sim d_s^{\pi', u_{\pi'}}} \langle Q_r^{\pi}, \pi' - \pi \rangle_{s'}$$

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## ▷ Global Convergence

$$\mathbb{E}_{s' \sim d_s^{\pi^*, u_{\pi}}} \left[ \langle Q_r^{\pi}, \pi - \pi^* \rangle_{s'} \right] \geq (1-\gamma)\left( V_r^{\pi}(s) - V_r^{\pi^*}(s) \right)$$

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▷ **Global Convergence**

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▷ $Q_r^{\pi}$ bears great similarities of subgradients for convex problems

# Robust Policy Mirror Descent: Deterministic Setting

# Convergence Characterization

**Theorem**

*Let* $M = \sup_{u \in \mathcal{U}} \|d_\rho^{\pi^*,u}/\rho\|_\infty$ *and* $M' = \sup_{u,u' \in \mathcal{U}} \|d_\rho^{\pi^*,u}/d_\rho^{\pi^*,u'}\|_\infty$*. In RPMD, choosing* $\eta_k \geq \eta_{k-1} \left(1 - \frac{1-\gamma}{M}\right)^{-1} M'$ *yields*

$$f_\rho(\pi_k) - f_\rho(\pi^*) \leq \left(1 - \frac{1-\gamma}{M}\right)^k \cdot \underbrace{\mathcal{O}(1)}_{\text{from initialization}}$$

1. First linear rate for first-order policy based method

# Convergence Characterization

**Theorem**

*Let* $M = \sup_{u \in \mathcal{U}} \|d_\rho^{\pi^*,u}/\rho\|_\infty$ *and* $M' = \sup_{u,u' \in \mathcal{U}} \|d_\rho^{\pi^*,u}/d_\rho^{\pi^*,u'}\|_\infty$*. In RPMD, choosing* $\eta_k \geq \eta_{k-1}\left(1 - \frac{1-\gamma}{M}\right)^{-1} M'$ *yields*

$$f_\rho(\pi_k) - f_\rho(\pi^*) \leq \left(1 - \frac{1-\gamma}{M}\right)^k \cdot \underbrace{\mathcal{O}(1)}_{\text{from initialization}}$$

1. First linear rate for first-order policy based method
2. Subsumes the special case of non-robust MDPs

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2. Subsumes the special case of non-robust MDPs

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3. Unclear whether dependence on $M$ is tight
   - Appears also for non-robust MDP with linear rate
   - Seems removable with a sublinear rate

# Robust Policy Mirror Descent: Stochastic Setting

# Stochastic Robust Policy Mirror Descent

**Algorithm** SRPMD update: $\pi_k \to \pi_{k+1}$

**Input**: Evaluate $\widehat{Q}_r^{\pi_k,\xi_k} \approx Q_r^{\pi_k}$

**Update**: For every state $s \in \mathcal{S}$:

$$\pi_{k+1}(\cdot|s) = \mathrm{argmin}_{p \in \Delta_{\mathcal{A}}} \eta_k \langle Q_r^{\pi_k,\xi_k}(s,\cdot), p \rangle + \mathcal{D}_{\pi_k}^p(s)$$

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- Converges up to the noise level
- Need to interact with $\mathbb{P}_\mathrm{N}$ to learn robust Q-function

# Learning the Robust Q-function

**Exploiting Access to $\mathbb{P}_{\mathrm{N}}$**

**Algorithm** Robust Temporal Difference Learning: $\pi \to Q_r^{\pi,\xi}$

**for** $t = 0, 1, \ldots$ **do**

Collect $s_{t+1} \sim \mathbb{P}_{\mathrm{N}}(\cdot|s_t, a_t)$, and make action $a_{t+1} \sim \pi(\cdot|s_{t+1})$

Update:

$$\begin{aligned}\theta_{t+1} = \theta_t + \alpha_t \big[ & c(s_t, a_t) + \gamma\theta_t(s_{t+1}, a_{t+1}) \\ & + \sigma_{\mathcal{U}_{s_t,a_t}}(M(\pi, \theta_t)) - \theta_t(s_t, a_t)\big] e(s_t, a_t)\end{aligned}$$

**end for**

- $\sigma_X(\cdot)$ is the support function of $X$
- $[M(\pi, x)](s) = \sum_{a\in\mathcal{A}} \pi(a|s)x(s,a)$ for $s \in \mathcal{S}$

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- When $\mathcal{U} = \{\mathbf{0}\}$, reduces to standard TD
- Can be easily adapted for $\epsilon$-contamination model
  - Unbiased robust Bellman evaluation operator is available

# Sample Complexity of RTD and SRPMD

▷ **Sample complexity of Robust TD**

**Proposition**

For any $\epsilon > 0$, with properly chosen $\alpha$, the RTD method needs at most

$$T = \widetilde{\mathcal{O}}\left(\frac{\log^2(1/\epsilon)}{(1-\gamma)^5 \nu_{\min}^3 \epsilon^2}\right)$$

iterations to find an estimate $\theta_T$ satisfying $\mathbb{E}_\xi \|\theta_T - Q_r^\pi\|_\infty \leq \epsilon$.

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▷ **Sample complexity of SRPMD**

**Theorem**

*With the same stepsize chosen as before, total number of samples required by SRPMD for finding an $\epsilon$-optimal policy can be bounded by*

$$\widetilde{\mathcal{O}}\left(\frac{M^3 \log^2\left(4M/(\epsilon(1-\gamma)^2)\right)}{(1-\gamma)^{10} \nu_{\min}^3 \epsilon^2}\right).$$

- We believe the dependence on $(1-\gamma)^{-1}$ can be improved

# Robust Policy Mirror Descent: (Linear) Function Approximation

# Preview of Linear Approximation

▷ **The essential target:** Find $\theta^\pi$ so that

$$\|\underbrace{\phi(\cdot,\cdot)^\top\theta^\pi}_{Q^\pi_{\theta^\pi}} - Q^\pi_r(\cdot,\cdot)\|_\infty$$

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**Isn't linear function approximation easy?**

1. Fixed-point (contraction) based:

$$Q^\pi_\theta = \Pi_{\phi,\nu}\mathcal{T}^\pi Q^\pi_\theta \;\rightarrow\; \theta^\pi$$

   - $\mathcal{T}^\pi$ – Robust Bellman operator of $Q^\pi_r$
   - $\Pi_{\phi,\nu}$ – the projection onto $\mathrm{span}(\Psi)$ in $\|\cdot\|_\nu$
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2. Minimize Bellman residual:

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   - Easily combined and nonlinear approximations (e.g., NNs)

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**Current Development**

No assumption-free convergent method for robust policy evaluation even in the deterministic setting

# Robust Evaluation as Policy Optimization

▷ **MDP of Nature:**

- State space: $\mathcal{S} \times \mathcal{A}$
- Action space: $\mathcal{U}_{s,a}$ for each $(s, a)$
- Transition: transition of $\{(s_t, a_t)\}$ generated by $\pi$ deployed in $\mathbb{P}_u^\pi$, where $u$ is determined by nature's policy
- Cost: $-c(s, a)$

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**Question: can we optimize nature's MDP efficiently?**

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**Question: can we optimize nature's MDP efficiently?**

Yes, $\mathcal{O}(1/\epsilon^2)$ sample suffices, even with linear approximation.

Also can be incorporated with NNs.

## Summary

1. RPMD for robust MDP with $(s, a)$-rectangular ambiguity
   - Simple implementation
   - Subsumes planning of non-robust MDP
2. Deterministic setting: $\mathcal{O}(\log(1/\epsilon))$ iterations
3. Stochastic setting:
   - Convergence up to noise level
   - $\widetilde{\mathcal{O}}(1/\epsilon^2)$ sample complexity
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   - Sample limit of policy gradients for robust MDP
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## Reference


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